

Domain wall roughening in dipolar films in the presence of disorder

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We derive a low-energy Hamiltonian for the elastic energy of a Néel domain wall in a thin film with in-plane magnetization, where we consider the contribution of the long-range dipolar interaction beyond the quadratic approximation. We show that such a Hamiltonian is analogous to the Hamiltonian of a one-dimensional polaron in an external random potential. We use a replica variational method to compute the roughening exponent of the domain wall for the case of two-dimensional dipolar interactions.

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I. INTRODUCTION

The roughening properties of elastic manifolds in the presence of quenched disorder is a well studied problem. The subject of study is usually the asymptotic (long-distance) regime in which one wishes to compute the roughening exponent ζ of the D dimensional elastic manifold [described by a displacement field $\phi(\mathbf{x})$] defined by

$$\overline{[\phi(\mathbf{x}) - \phi(\mathbf{x}')]^2} = |\mathbf{x} - \mathbf{x}'|^{2\zeta}, \quad (1)$$

where \mathbf{x} is the position vector in the D -dimensional configuration space and where the brackets represent an average over thermal fluctuations and the overline an average over the realizations of the quenched disorder. Obviously, the study of the crossover between short distances, where the thermal fluctuations dominate and the long-distance behavior, which is determined by the disorder, is also of interest. Among the methods applied to the study of this problem are the use of Imry-Ma type of arguments [1–3], the mapping of one-dimensional interfaces in two dimensions to the noisy Burgers equation [4], which yields an exact result for the roughening exponent [5,6], variational approaches involving replica averaging and replica symmetry breaking [7–9], and functional renormalization group (RG) calculations [10,11]. The role played by long-range interactions does not seem to be so well understood, although it follows from the one-loop functional RG results of Emig and Nattermann [11] that in the important case of a magnetic domain wall in the presence of long-range dipolar interactions, one simply needs to replace the expansion parameter $\epsilon = 4 - D$ by $\epsilon = 3(3 - D)/2$ (where D is the dimension of the manifold) in the expressions for the critical exponents obtained in the absence of dipolar interactions, in order to account for the presence of these interactions [12]. In particular, if one applies this result to the case of a line domain in a thin magnetic film ($D = 1$), the effective expansion parameter is $\epsilon = 3$ in both cases, which means that in this case and to one-loop order, the critical exponents obtained are the same in the absence or

presence of dipolar interactions. Although one expects the RG calculations to provide a qualitative understanding at low dimensions compared to the upper critical dimension ($D = 3$ in this case), it is still questionable to what extent one can perform such analytical continuation using just the one-loop results. Furthermore, the results of Emig and Nattermann were based on a series expansion of the dipolar energy [13,14] which meant that, in the absence of disorder, the Hamiltonian describing the low-energy degrees of freedom of the domain wall was quadratic in the domain-wall displacement field, i.e., the random potential is the only source of nonlinearity in the problem. One still needs to justify that such approximation is sufficient [15].

Having in mind such difficulties, we wish to discuss the properties of a one-dimensional domain wall in a two-dimensional ferromagnetic film, in the presence of dipolar interactions and a short-range correlated random field, where the dipolar interaction is treated beyond the quadratic approximation. We wish to check whether it is still possible to map such a problem to a (modified) Burgers equation, as was done by Huse *et al.* [5] for the case of random-bond disorder and by Zhang [6] for the case of random-field disorder, in the absence of long-range interactions. Such a mapping was achieved because one can map the line domain to the world line of a quantum particle in $1 + 1$ dimensions. The free energy of the line domain can then be shown to obey a Schrödinger equation in imaginary time in the presence of a random potential. A Cole-Hopf transformation can then be used to convert this equation into a noisy Burgers equation with additive noise, for which the stationary distribution is known in one-dimension [16]. This allows one to compute the roughening exponent. In our case, the presence of the dipolar interaction does not allow one to write the energy of the domain wall as an action for a single quantum particle, but instead we obtain an action for a quantum particle that interacts with an annealed “vector potential,” whose propagator is determined by the dipolar propagator. In $1 + 1$ dimensions, a gauge transformation allows us to substitute the vector potential by a scalar one, which can then be integrated out, leaving us with an action similar to the action of a polaron as considered by Feynman [17,18], i.e., the dipolar interaction introduces a “self-retarded” interaction of the particle with itself. Despite the fact that such action cannot be integrated exactly, we can still apply a variational method in the spirit

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of Feynman [7–9], in which one introduces replicas of the system in order to average the free energy of the domain wall over the realizations of the random field. Using the hierarchical replica symmetry breaking (RSB) ansatz of Parisi [19], we were able to derive a set of nonlinear equations whose self-consistent solution yields the long-distance physics of the problem within the realm of this approximation. With this method, we are able to treat two different situations. The one of a thin ferromagnetic film grown on a nonmagnetic substrate, in which the dipolar interaction has the usual three-dimensional form, and the one of a thin ferromagnetic film grown between two type I superconductors, i.e., the case of a superconductor/ferromagnet/superconductor (SC/FM/SC) heterostructure. We have shown elsewhere [20] that in this case the dipolar interaction has a two-dimensional form at small wave vectors, coming from the renormalization of the magnetic energy due to the Meissner effect in the superconductors. Although we were unable to obtain an analytic solution of the variational nonlinear equations, we were able to justify that in the case of two-dimensional dipolar interactions (i.e., for the SC/FM/SC heterostructure), the quadratic approximation is indeed adequate. This case is not only easier to treat analytically, it is also more interesting than the case of three-dimensional dipolar interactions, for which one expects the dipolar interaction to be irrelevant (see above). Therefore, we restrict our discussion to the case of two-dimensional dipolar interactions in this paper. Furthermore, we can also show that in this case, if one works within the quadratic approximation, the elastic energy of the domain wall has the same analytical form as the energy of a domain wall in a ferroelastic material as studied by Kolomeiskii *et al.* [12] and also of the energy of a liquid-gas interface in a disordered solid, as studied by Hazareesing and Mézard [9]. In both cases, one obtains, in two dimensions, a value of 1/3 for the roughening exponent. This is indeed the value that we obtain for the roughening exponent in our problem. It should be nevertheless noticed that these three problems, though mathematically similar have a different physical origin and motivation.

The structure of this paper is as follows. In Sec. II, we derive a low-energy Hamiltonian describing the Néel domain wall elastic degrees of freedom. This Hamiltonian can be interpreted as the action in imaginary time of a quantum particle in 1+1 dimensions. In Sec. III, we use the replica trick to average over the random field realizations, which generates a many particle nonquadratic action. In Sec. IV, we introduce a generalization of the variational ansatz of Mézard and Parisi which allows us to consider the effect of the short-range correlated disorder and of the nonlocal dipolar term in the many particle action. One obtains a set of nonlinear equations that have to be solved self-consistently. In Sec. V, we perform the analysis of these equations in the particular case of the hierarchical replica symmetry breaking solution of Parisi and we discuss the validity of the quadratic approximation for the dipolar term in these equations, which permits to determine the roughening exponent of the domain wall. Finally, in Sec. VI, we present our conclusions.

II. ENERGY OF A SINGLE DISTORTED DOMAIN WALL IN A TWO-DIMENSIONAL XY MODEL WITH DIPOLAR INTERACTIONS AND IN A RANDOM FIELD

We consider a Néel (i.e., 180°) domain wall in a thin two-dimensional ferromagnetic film in the presence of dipolar interactions and a short-range correlated quenched random field. The magnetization is in the plane of the film. In the straight domain configuration, the wall is oriented along the y axis and the center of the wall is located at $x=0$. The thermal and random field fluctuations give rise to deviations of the domain wall from its straight configuration, i.e., at a given position y , the center of the domain is displaced from $x=0$ to $x(y)$ (we neglect overhangs). In this case, Eq. (1) reduces to

$$\overline{[x(y) - x(y')]^2} = |y - y'|^{2\zeta}. \quad (2)$$

The distance $|y - y'|$ is considered to be very large with respect to the lattice spacing and to the length over which the random field is correlated (see below). In the absence of dipolar interactions, one can, as stated above, through a mapping to a Schrödinger equation [6], show that $\zeta = 1$.

In order to determine the elastic energy of the domain wall, we consider the underlying system of ferromagnetic spins to be in a square lattice with lattice constant a and overall dimensions $L_x \times L_y$. The system has antiperiodic boundary conditions along the x direction (in order that there is a single domain wall in the system) and periodic boundary conditions along the y direction and is at finite temperature T . Such system is described by a (classical) XY model Hamiltonian, with dipolar interactions and in a random field,

$$\mathcal{H} = -\frac{1}{2} \sum_{\langle i,j \rangle} J_{ij} \mathbf{S}_i \cdot \mathbf{S}_j - \mu \sum_i \mathbf{h}_i \cdot \mathbf{S}_i + \frac{G}{2} \sum_{i \neq j} \left(\frac{\mathbf{S}_i \cdot \mathbf{S}_j}{|\mathbf{R}_i - \mathbf{R}_j|^n} - \frac{n(\mathbf{R}_i - \mathbf{R}_j) \cdot \mathbf{S}_i(\mathbf{R}_i - \mathbf{R}_j) \cdot \mathbf{S}_j}{|\mathbf{R}_i - \mathbf{R}_j|^{n+2}} \right) - D \sum_i (S_i^y)^2. \quad (3)$$

For the case of two-dimensional dipolar interactions (i.e., for the SC/FM/SC heterostructure), the constants μ and G are given in the mks system by $\mu = \mu_0 \mu_B g_L$ and $G = (\mu_0/4\pi \mu_r \lambda_L)(g_L \mu_B)^2$, where μ_0 is the magnetic permittivity of the vacuum, μ_B the Bohr magneton, and g_L the Landé factor of the spin system. The constants λ_L and μ_r are, respectively, the London penetration depth and relative magnetic permittivity of the superconductor. In this case, $n = 2$. In the case of three-dimensional dipolar interactions, which we will not consider, $G = (\mu_0/4\pi)(\mu_B g_L)^2$ and n is equal to 3. The quantities J_{ij} and D have the dimensions of energy [21]. The field \mathbf{h}_i is a random field with Gaussian correlations, i.e., $\overline{h_i^\alpha h_j^\beta} = \Delta_{ij} \delta^{\alpha\beta}$. In order to stabilize the domain wall, we have added to the Hamiltonian above an easy-axis anisotropy term, along the y axis, represented by the last term in Eq. (3).

For a straight Néel domain wall, the spin configuration has the form

$$S_i^x = S \cos \phi(x_i), \quad (4)$$

$$S_i^y = S \sin \phi(x_i), \quad (5)$$

where i is the site label and where $\phi(x)$ goes from $-\pi/2$ when $x \rightarrow -L_x/2$ to $\pi/2$ when $x \rightarrow L_x/2$, with S being the value of the spin. The function $\phi(x)$ changes its value from $-\pi/2$ to $\pi/2$ in a region of width w around $x=0$. Since w is typically of the order of tens of nanometers [22] and we are only interested in the long-distance properties of the domain wall, we can write $d\phi(x)/dx = \pi\delta(x)$. If the center of the domain wall is displaced from $x_i=0$ to $x_i=x(y_i)$, the spin configuration becomes

$$S_i^x = S \cos \phi(x_i - x(y_i)), \quad (6)$$

$$S_i^y = S \sin \phi(x_i - x(y_i)). \quad (7)$$

We wish to compute the partition function $\mathcal{Z} = \text{Tr}(e^{-\delta\mathcal{H}/k_B T})$, where $\delta\mathcal{H} = \delta\mathcal{H}_{exc} + \delta\mathcal{H}_{rf} + \delta\mathcal{H}_{dip}$ is the energy difference between a configuration described by the displacement $x(y)$ and the straight domain configuration. We define the discrete Fourier transforms

$$S_q^\alpha = \sum_{R_i} S_i^\alpha e^{-iq \cdot R_i}, \quad (8)$$

$$x(q_y) = \sum_{y_i} x(y_i) e^{-iq_y y_i}, \quad (9)$$

where \mathbf{q} belongs to the first Brillouin zone. We will approximate the sums over i by integrals, according to the formulas,

$$\sum_{R_i} \approx \frac{1}{a^2} \int d^2x, \quad \sum_{y_i} \approx \frac{1}{a} \int dy, \quad (10)$$

valid for a square lattice. It can be shown that the (positive) contribution to the exchange energy coming from the domain-wall deformation is given to second order in the displacement field, by [13]

$$\begin{aligned} \delta\mathcal{H}_{exc} &= -\frac{\pi^2 S^2}{2Na^4} \sum_q [J(q_x, q_y) - J(q_x, 0)] x(q_y) x(-q_y) \\ &\approx \frac{\pi^2 JS^2}{2L_y a} \sum_{q_y} q_y^2 x(q_y) x(-q_y), \end{aligned} \quad (11)$$

where $N = L_x L_y / a^2$ is the total number of sites and where we have expanded $J(\mathbf{q}) \approx J_0 - Ja^2 q^2$. In real space, Eq. (11) reads

$$\delta\mathcal{H}_{exc} = \frac{1}{2} \sigma \int_{-L_y/2}^{L_y/2} dy \left(\frac{dx}{dy} \right)^2, \quad (12)$$

which has the same form as the kinetic part of the action of a particle of mass σ in one dimension, with $\sigma = \pi^2 JS^2/a$ being the line tension of the domain wall. The contribution coming from the random field can be shown to be equal to [23]

$$\delta\mathcal{H}_{rf} = \tilde{\mu} \int_{-L_y/2}^{L_y/2} dy \int_0^{x(y)} dx \tilde{h}^y(x, y), \quad (13)$$

where $\tilde{\mu} = \mu_0 \mu_B g_L \pi S/a$ and $\tilde{h}^y = h^y/a$ [24]. This formula can be made plausible if we consider instead a constant field along y . It can also be easily seen from Eq. (11) that the easy-axis anisotropy term does not contribute to the deformation energy of the domain wall.

Since we wish to compute the partition function, we have to consider the factor $e^{-\mathcal{H}_{dip}/k_B T}$, where \mathcal{H}_{dip} is given by

$$\mathcal{H}_{dip} = \frac{g}{2V} \sum_q \mathcal{F}(\mathbf{q}) (\mathbf{q} \cdot \mathbf{S}_q) (\mathbf{q} \cdot \mathbf{S}_{-q}), \quad (14)$$

where $V = L_x L_y$ is the area of the sample. For two-dimensional dipolar interactions, $g = (\mu_0/2\mu_r \lambda_L)(g_L \mu_B)^2$ and $\mathcal{F}(\mathbf{q}) = 1/q^2$ [20]. For three-dimensional dipolar interactions, $g = (\mu_0/2)(g_L \mu_B)^2$ and $\mathcal{F}(\mathbf{q}) = 1/q$. Thus, by considering the different dipolar kernels $\mathcal{F}(\mathbf{q})$, one can treat the two different cases.

Using a Hubbard-Stratonovich transformation, we write $e^{-\mathcal{H}_{dip}/k_B T}$ as

$$\begin{aligned} e^{-\mathcal{H}_{dip}/k_B T} &= \mathcal{N} \int \mathcal{D}A_q \exp \left(-\frac{1}{2Vk_B T} \sum_q A_q \mathcal{F}^{-1}(\mathbf{q}) A_{-q} \right. \\ &\quad \left. + \frac{\sqrt{g}}{Vk_B T} \sum_q (\mathbf{q} \cdot \mathbf{S}_q) A_{-q} \right), \end{aligned} \quad (15)$$

where \mathcal{N} is a normalization factor. We can now show that

$$\mathbf{q} \cdot \mathbf{S}_q = \frac{i\pi S}{a^2} \int_{-L_y/2}^{L_y/2} dy e^{-iq_x x(y) - iq_y y} \frac{dx}{dy}, \quad (16)$$

from which it follows that

$$\frac{1}{V} \sum_q (\mathbf{q} \cdot \mathbf{S}_q) A_{-q} = \frac{i\pi S}{a^2} \int_{-L_y/2}^{L_y/2} dy A(x(y), y) \frac{dx}{dy}, \quad (17)$$

and one can see that there is no contribution to the dipolar energy when $x(y) = \text{const}$. The partition function of the domain wall's degrees of freedom involves the functional integration of the Boltzmann factor $e^{-\delta\mathcal{H}/k_B T}$, over the space of functions $x(y)$.

We shall now make a change in our notation. We will write

$$\begin{aligned} y \rightarrow \tau, \quad x(y) \rightarrow x(\tau), \quad \frac{dx}{dy} \rightarrow \dot{x}(\tau), \\ q_y \rightarrow \omega_m, \quad e = \frac{\pi S \sqrt{g}}{a^2}. \end{aligned} \quad (18)$$

The subscript m stands for the fact that the wave vectors ω_m are discrete, due to the periodic boundary conditions along

$\tau(y)$. If we use this notation in the formulas above, we have, collecting all the factors, the following expression for the partition function:

$$\mathcal{Z} = \mathcal{N} \int \mathcal{D}x(\tau) \mathcal{D}A(q_x, \omega_m) e^{-S/k_B T}, \quad (19)$$

where the action S is given by

$$\begin{aligned} S = & \frac{1}{2} \sigma \int_{-L_y/2}^{L_y/2} d\tau \dot{x}^2(\tau) + \tilde{\mu} \int_{-L_y/2}^{L_y/2} d\tau \int_0^{x(\tau)} dx \tilde{h}^y(x, \tau) \\ & - ie \int_{-L_y/2}^{L_y/2} d\tau A(x(\tau), \tau) \dot{x}(\tau) \\ & + \frac{1}{2V} \sum_{q_x, \omega_m} A(q_x, \omega_m) \mathcal{F}^{-1}(q_x, \omega_m) A(-q_x, -\omega_m). \end{aligned} \quad (20)$$

This is the action of a single quantum particle in 1+1 dimensions, with “mass” σ and “charge” e , at “inverse temperature” L_y and with a “Planck constant” equal to $k_B T$. Such particle is in an external random potential $V(x, \tau) = \tilde{\mu} \int_0^x dx' \tilde{h}^y(x', \tau)$ and interacts with a “vector potential” $A(q_x, \omega_m)$ [25] characterized by a propagator equal to $\mathcal{F}(q_x, \omega_m)$. This result belongs to the well known class of mappings of two dimensional problems of classical statistical mechanics to 1+1 dimensional quantum problems. Now, since one has periodic boundary conditions along the $\tau(y)$ axis, the following identity holds:

$$\int_{-L_y/2}^{L_y/2} d\tau A(x(\tau), \tau) \dot{x}(\tau) = - \int_{-L_y/2}^{L_y/2} d\tau \int_0^{x(\tau)} dx' \frac{\partial A(x', \tau)}{\partial \tau}, \quad (21)$$

which corresponds to a gauge transformation of the action [26]. The use of this identity eliminates the dependence on the “velocity” $\dot{x}(\tau)$ in all terms of Eq. (20) except the first one. The action S now reads

$$\begin{aligned} S = & \frac{1}{2} \sigma \int_{-L_y/2}^{L_y/2} d\tau \dot{x}^2(\tau) + \int_{-L_y/2}^{L_y/2} d\tau V(x(\tau), \tau) \\ & + ie \int_{-L_y/2}^{L_y/2} d\tau \int_0^{x(\tau)} dx' \frac{\partial A(x', \tau)}{\partial \tau} \\ & + \frac{1}{2V} \sum_{q_x, \omega_m} A(q_x, \omega_m) \mathcal{F}^{-1}(q_x, \omega_m) A(-q_x, -\omega_m). \end{aligned} \quad (22)$$

We can now integrate out the field $A(q_x, \omega_m)$, obtaining an action entirely in terms of $x(\tau)$ alone. We obtain

$$\begin{aligned} S = & \frac{1}{2} \sigma \int_{-L_y/2}^{L_y/2} d\tau \dot{x}^2(\tau) + \int_{-L_y/2}^{L_y/2} d\tau V(x(\tau), \tau) \\ & + \frac{e^2}{2V} \sum_{q_x, \omega_m} \frac{\omega_m^2}{q_x^2} \mathcal{F}(q_x, \omega_m) \int_{-L_y/2}^{L_y/2} d\tau \int_{-L_y/2}^{L_y/2} d\tau' \\ & \times \exp\{-iq_x[x(\tau) - x(\tau')] - i\omega_m(\tau - \tau')\}. \end{aligned} \quad (23)$$

This action can be used to compute the roughening exponent of the domain wall. Equation (23) also shows that, apart from the interaction with the τ dependent random potential $V(x, \tau)$, the action S is analogous to the action of a one-dimensional polaron, as written by Feynman [17,18].

However, in order to compute the roughening exponent, one has to determine the following quantity:

$$\overline{\langle (x(\tau) - x(\tau'))^2 \rangle} = \frac{\int \mathcal{D}x(\tau) e^{-S/k_B T} [x(\tau) - x(\tau')]^2}{\int \mathcal{D}x(\tau) e^{-S/k_B T}}, \quad (24)$$

which involves determining the average of the distribution of quenched disorder over a quotient. This type of averaging places the same problem as when one needs to average over the free energy of the system, i.e., one needs to determine the average $\ln \mathcal{Z}$, instead of simply averaging over the partition function \mathcal{Z} . This can be seen by including a source field coupled to $x(\tau)$ in \mathcal{Z} . The two-point correlation function appearing in Eq. (24) can be computed from $\ln \mathcal{Z}$ by differentiating it with respect to the sources and taking these sources to be zero.

III. THE AVERAGE OVER DISORDER: REPLICAS TRICK

We need to perform the average over the random field $\tilde{h}^y(q_x, \omega_m)$, which obeys a Gaussian distribution with a second moment equal to

$$\langle \tilde{h}^y(q_x, \omega_m) \tilde{h}^y(q'_x, \omega'_m) \rangle = \frac{N}{a^2} \Delta(q_x, \omega_m) \delta_{q_x + q'_x, 0} \delta_{\omega_m + \omega'_m, 0}. \quad (25)$$

In order to perform the averaging over disorder in the free energy, one introduces n distinct copies (replicas) of the system [27] and makes use of the identity

$$\ln \mathcal{Z} = \lim_{n \rightarrow 0} \frac{\mathcal{Z}^n - 1}{n}. \quad (26)$$

One performs the disorder average over \mathcal{Z}^n for integer n and makes the analytical continuation to $n \rightarrow 0$. One obtains

$$\overline{\mathcal{Z}^n} = \int \prod_a \mathcal{D}x_a(\tau) e^{-S_R/k_B T}, \quad (27)$$

where the “replicated” action S_R is given by

$$\begin{aligned}
 S_R = & \frac{1}{2} \sigma \sum_a \int_{-L_y/2}^{L_y/2} d\tau [\dot{x}_a^2(\tau) + \Omega_0^2 x_a^2(\tau)] \\
 & + \frac{\tilde{\mu}^2 n}{k_B T L_x} \sum_{a, q_x} \frac{\Delta(q_x, 0)}{q_x^2} \int_{-L_y/2}^{L_y/2} d\tau \cos(q_x x_a(\tau)) \\
 & + \frac{e^2}{2V} \sum_{a, q_x, \omega_m} \frac{\omega_m^2}{q_x^2} \mathcal{F}(q_x, \omega_m) \int_{-L_y/2}^{L_y/2} d\tau \int_{-L_y/2}^{L_y/2} d\tau' \\
 & \times \exp\{-i q_x [x_a(\tau) - x_a(\tau')] - i \omega_m (\tau - \tau')\} \\
 & - \frac{\tilde{\mu}^2}{k_B T V} \sum_{a, b, q_x, \omega_m} \frac{\Delta(q_x, \omega_m)}{q_x^2} \int_{-L_y/2}^{L_y/2} d\tau \int_{-L_y/2}^{L_y/2} d\tau' \\
 & \times \exp\{-i q_x [x_a(\tau) - x_b(\tau')] - i \omega_m (\tau - \tau')\}, \quad (28)
 \end{aligned}$$

and where the last term represents the inter-replica interaction, due to the presence of the random potential. We have included in S_R the effect of an applied field $\tilde{h}_{e_{xt}}^y(x) = (\sigma \Omega_0^2 / \tilde{\mu}) x$. This term gives rise to the second term in the action, which is a harmonic potential. The presence of this term is necessary to guarantee that S_R is bounded from below, given that the last term of Eq. (28) comes with a negative sign. Ω_0 can be set to 0 at the end of the calculations.

IV. A VARIATIONAL ANSATZ FOR THE FREE ENERGY: VARIATIONAL EQUATIONS

The action (28) cannot be integrated exactly. Therefore, we need to develop an approximation scheme. We shall follow Mézard and Parisi [7] and Goldschmidt [8]. We choose a trial quadratic action of the form

$$\begin{aligned}
 S_0[\eta_a(\tau)] = & \frac{1}{2} \sigma \sum_a \int_{-L_y/2}^{L_y/2} d\tau [\dot{x}_a^2(\tau) + \Omega_a^2 x_a^2(\tau)] \\
 & - \frac{1}{2} \sum_{a, b} \int_{-L_y/2}^{L_y/2} d\tau \int_{-L_y/2}^{L_y/2} d\tau' x_a(\tau) Q_{ab}(\tau - \tau') \\
 & \times x_b(\tau') - \sum_a \int_{-L_y/2}^{L_y/2} d\tau \eta_a(\tau) x_a(\tau), \quad (29)
 \end{aligned}$$

where we have included a source field $\eta_a(\tau)$, which is useful in the calculation of correlation functions. The Mézard-Parisi ansatz has been generalized by including a matrix function $Q_{ab}(\tau - \tau')$, with a nontrivial dependence on $\tau - \tau'$, necessary to take into account the effect of the nonlocal dipolar interaction. In the Mézard and Parisi case, $Q_{ab}(\tau - \tau') = Q_{ab} \delta(\tau - \tau')$, since they considered only short-range correlated disorder. Goldschmidt has considered several different forms for the kernel $Q_{ab}(\tau - \tau')$, all of which is nonlocal in $\tau - \tau'$, and which are able to account for the problem of a quantum particle in a disordered static potential and for the related problem of a polymer in a random potential with long-range correlated disorder (see below). Since $Q_{ab}(\tau) = Q_{ab}(\tau + L_y)$, we can represent $Q_{ab}(\tau)$ in terms of Matsubara modes $Q_{ab}(\tau) = 1/L_y \sum_m Q_{ab}^m e^{i \omega_m \tau}$. The values of Ω_a^2 and

Q_{ab}^m are to be chosen appropriately. We also demand S_0 to be translationally invariant when $\Omega_a \rightarrow 0$. This gives rise to the constraint

$$Q_{bb}^0 = - \sum_{a \neq b} Q_{ab}^0. \quad (30)$$

It can be shown that the normalized generating function for the correlation functions of such a quadratic action, is given by [28]

$$\begin{aligned}
 Z_N[\eta_a(\tau)] = & \exp\left(1/2(k_B T)^2 \sum_{ab} \int_{-L_y/2}^{L_y/2} d\tau \int_{-L_y/2}^{L_y/2} d\tau' \right. \\
 & \left. \times \eta_a(\tau) G_{ab}(\tau - \tau') \eta_b(\tau')\right), \quad (31)
 \end{aligned}$$

where $G_{ab}(\tau - \tau') = \langle x_a(\tau) x_b(\tau') \rangle$ is given in terms of its Matsubara modes, by

$$G_{ab}^m = k_B T [\sigma(\omega_m^2 \hat{I} + \hat{\Omega}^2) - \hat{Q}^m]_{ab}^{-1}, \quad (32)$$

where \hat{I} is the unit matrix in replica space and $\hat{\Omega}^2$ is the diagonal matrix with elements equal to Ω_a^2 . The free energy as computed from the trial action $S_0[0]$, with the sources $\eta_a(\tau)$ equal to zero, can be shown to be equal to

$$F_0 = C^t - \frac{k_B T}{2} \sum_{m, \mu} \ln\{(1/k_B T) [\sigma \omega_m^2 + (\sigma \hat{\Omega}^2 - \hat{Q}^m)_\mu]\}, \quad (33)$$

where μ indicates the basis in which the matrix $\sigma \hat{\Omega}^2 - \hat{Q}^m$ is diagonal. One now uses the Feynman inequality for $F = -k_B T \ln \bar{Z}^n$, which states that

$$F \leq F_{var} = F_0 + \langle S_R - S_0[0] \rangle_0, \quad (34)$$

where the average on the right-hand side of Eq. (34) is over the trial quadratic action $S_0[0]$. Note that $\mathcal{F} = -k_B T \ln \bar{Z} = \lim_{n \rightarrow 0} F/n$. One obtains for F_{var} the expression

$$\begin{aligned}
 \frac{F_{var}}{L_y} = & \tilde{C}^t - \frac{k_B T}{2L_y} \sum_{m, \mu} \ln\{(1/k_B T) [\sigma \omega_m^2 + (\sigma \hat{\Omega}^2 - \hat{Q}^m)_\mu]\} \\
 & + \frac{\sigma}{2L_y} \sum_{a, m} (\Omega_0^2 - \Omega_a^2) \{ (1/k_B T) [\sigma \omega_m^2 \hat{I} + (\sigma \hat{\Omega}^2 \\
 & - \hat{Q}^m)] \}_{aa} + \frac{1}{2L_y} \sum_{a, b, m} Q_{ab}^m \{ (1/k_B T) [\sigma \omega_m^2 \hat{I} + (\sigma \hat{\Omega}^2 \\
 & - \hat{Q}^m)] \}_{ba} + \frac{\tilde{\mu}^2 n}{k_B T L_x} \sum_{a, q_x} \frac{\Delta(q_x, 0)}{q_x^2} e^{-(1/2) q_x^2 G_{aa}(0)} \\
 & + \frac{1}{2V} \sum_{a, b, q_x, \omega} \int_{-L_y/2}^{L_y/2} d\tau \left(\frac{e^2 \omega^2}{q_x^2} \mathcal{F}(q_x, \omega) \delta_{ab} \right)
 \end{aligned}$$

$$\begin{aligned} & -\frac{\tilde{\mu}^2}{k_B T} \frac{\Delta(q_x, \omega)}{q_x^2} \Big) \exp \left\{ -i\omega\tau - \frac{1}{2} q_x^2 [G_{aa}(0) \right. \\ & \left. + G_{bb}(0) - 2G_{ab}(\tau)] \right\}, \end{aligned} \quad (35)$$

where the following identities, valid for a quadratic action at zero sources,

$$\langle e^{-iq_x x_a(\tau)} \rangle_0 = e^{-(1/2)q_x^2 G_{aa}(0)}$$

and

$$\begin{aligned} \langle e^{-iq_x [x_a(\tau) - x_b(0)]} \rangle_0 = & \exp \left\{ -\frac{1}{2} q_x^2 [G_{aa}(0) \right. \\ & \left. + G_{bb}(0) - 2G_{ab}(\tau)] \right\} \end{aligned}$$

were used. Following Hazareesing and Mézard [9], we choose the random field to have anisotropic correlations, $\Delta(q_x, \omega_m) = \Delta e^{-1/2 q_x^2 \mathcal{B}^2}$, with correlation length \mathcal{B} along x [29,30] and we choose the two-dimensional form of the dipolar kernel, i.e., $\mathcal{F}(q_x, \omega_m) = 1/(q_x^2 + \omega_m^2)$. We have to minimize Eq. (35) with respect to the free parameters Q_{ab}^m and Ω_a^2 . Such procedure gives the following self-consistent equations:

$$\Omega_a^2 = \Omega_0^2 - \frac{\tilde{\mu}^2 \Delta n}{\sqrt{2\pi k_B T}} [\mathcal{B}^2 + G_{aa}(0)]^{-1/2}, \quad (36)$$

$$Q_{ab}^m = Q_{ab}^0 + f_a^m \delta_{ab}, \quad m \neq 0 \quad (37)$$

with

$$Q_{ab}^0 = \frac{\tilde{\mu}^2 \Delta}{\sqrt{2\pi k_B T}} [\mathcal{B}^2 + G_{aa}(0) + G_{bb}(0) - 2G_{ab}(0)]^{-1/2}, \quad (38)$$

($a \neq b$),

$$\begin{aligned} f_a^m = & -e^2 \int_0^{L_y/2} d\tau [1 - \cos(\omega_m \tau)] \left[\frac{1}{2\pi \gamma_{aa}(\tau)} \right. \\ & \left. - \frac{\tau}{4\sqrt{\pi} \gamma_{aa}^{3/2}(\tau)} e^{\tau^2/4\gamma_{aa}(\tau)} \operatorname{erfc} \left(\frac{\tau}{2\gamma_{aa}^{1/2}(\tau)} \right) \right], \end{aligned} \quad (39)$$

with Q_{bb}^0 given by Eq. (30) and where $\gamma_{aa}(\tau) = G_{aa}(0) - G_{aa}(\tau)$ [31]. We have approximated the sum over q_x by an integral, a procedure that becomes exact in the thermodynamic limit, and $\operatorname{erfc}(x)$ is the complementary error function. These equations constitute a closed nonlinear system that has to be solved self-consistently in the limit $n \rightarrow 0$. We consider the solution of these equations in this limit in the next section.

V. ANALYSIS OF THE VARIATIONAL EQUATIONS FOR A TWO-DIMENSIONAL DIPOLAR INTERACTION: HIERARCHICAL REPLICA SYMMETRY BREAKING

One needs to take the limit $n \rightarrow 0$ in the self-consistent equations derived above. This implies dealing with the matrix structure of the equations in this limit. One cannot do this in general. However, there is a particular parametrization of these matrices, due to Parisi [19], which is sufficiently general for our purposes. This is the so called hierarchical RSB. The diagonal elements of the matrix Q_{ab} (with dimensions $n \times n$ where n is arbitrarily large) are taken to be all equal to \tilde{q} and the off-diagonal elements are taken to be equal to q_0 . Q_{ab} is then partitioned in block-diagonal submatrices, the elements of Q_{ab} outside these blocks keeping the value q_0 and the elements inside the blocks taking a new value q_1 (except the diagonal elements that keep their value \tilde{q}). The procedure is repeated in an equal form for every diagonal block submatrix, the off-diagonal elements of these submatrices keeping the value q_1 and the elements inside the diagonal subblocks taking the value q_2 . This procedure is repeated “*ad infinitum*.” The multiplication rules obeyed by these hierarchical matrices are analytically continued to $n \rightarrow 0$. The off-diagonal elements of Q_{ab} are then parametrized by a function $q(u), u \in [0,1]$ and the diagonal elements are parametrized by the number \tilde{q} . The constraint (30) takes the form

$$\tilde{q} = \int_0^1 du q(u), \quad (40)$$

and one can show that the matrices G_{ab}^m are also parametrized by a function $g^m(u)$ and by \tilde{g}^m that parametrizes the diagonal elements. In the thermodynamic limit $L_x = L_y \rightarrow \infty$, $\omega_m \rightarrow \omega$, $\tilde{g}^m \rightarrow \tilde{g}(\omega)$, and $g^m(u) \rightarrow g(\omega, u)$ and one gets, using Eqs. (36)–(39), the following relations, where $\tilde{g}(\tau)$ [$g(\tau, u)$] is the Fourier transform of $\tilde{g}(\omega)$ [$g(\omega, u)$],

$$\hat{\Omega}^2 = \Omega_0^2 - \frac{\tilde{\mu}^2 \Delta n}{\sqrt{2\pi k_B T}} [\mathcal{B}^2 + \tilde{g}(\tau=0)]^{-1/2}, \quad (41)$$

$$q(u) = \frac{\tilde{\mu}^2 \Delta}{\sqrt{2\pi k_B T}} \{ \mathcal{B}^2 + 2[\tilde{g}(\tau=0) - g(\tau=0, u)] \}^{-1/2}, \quad (42)$$

$$\begin{aligned} f(\omega) = & -e^2 \int_0^\infty d\tau [1 - \cos(\omega \tau)] \left[\frac{1}{2\pi \gamma(\tau)} \right. \\ & \left. - \frac{\tau}{4\sqrt{\pi} \gamma^{3/2}(\tau)} e^{\tau^2/4\gamma(\tau)} \operatorname{erfc} \left(\frac{\tau}{2\gamma^{1/2}(\tau)} \right) \right], \end{aligned} \quad (43)$$

and where $\gamma(\tau) = \tilde{g}(\tau=0) - \tilde{g}(\tau)$. One sees from Eq. (41) that when $n \rightarrow 0$, $\hat{\Omega}^2 = \Omega_0^2$. The functions $\tilde{g}(\omega)$ and $g(\omega, u)$ are in turn given by

$$\tilde{g}(\omega) = \frac{k_B T}{\sigma(\omega^2 + \Omega_0^2) - f(\omega)} \left(1 + \frac{q(0)}{\sigma(\omega^2 + \Omega_0^2) - f(\omega)} + \int_0^1 \frac{dv}{v^2} \frac{[q](v)}{\sigma(\omega^2 + \Omega_0^2) - f(\omega) + [q](v)} \right), \quad (44)$$

$$g(\omega, u) = \frac{k_B T}{\sigma(\omega^2 + \Omega_0^2) - f(\omega)} \left(\frac{q(0)}{\sigma(\omega^2 + \Omega_0^2) - f(\omega)} + \int_0^u \frac{dv}{v^2} \frac{[q](v)}{\sigma(\omega^2 + \Omega_0^2) - f(\omega) + [q](v)} + \frac{[q](u)}{u\{\sigma(\omega^2 + \Omega_0^2) - f(\omega) + [q](u)\}} \right), \quad (45)$$

where $[q](u) = uq(u) - \int_0^u dv q(v)$. Substituting Eqs. (44) and (45) in Eq. (42) and differentiating with respect to u , one obtains the equations

$$q'(u) = 0, \quad (46)$$

or

$$q(u) = \left(\frac{\tilde{\mu}^2 \Delta}{\sqrt{2\pi} (k_B T)^{3/2}} \right)^{2/3} \times \left(\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{1}{\{\sigma(\omega^2 + \Omega_0^2) - f(\omega) + [q](u)\}^2} \right)^{-1/3}.$$

If the first equation is valid in the whole interval $u \in [0, 1]$, its solution is simply $q(u) = \text{const}$, which is the replica symmetric solution. The second equation is easily solvable if $f(\omega) = 0$ (i.e., $e = 0$) and the solution was discussed by Mézard and Parisi [7]. Also, one has

$$\overline{[x(\tau) - x(\tau')]^2} = \lim_{n \rightarrow 0} \frac{2}{n} \sum_a [G_{aa}(0) - G_{aa}(\tau - \tau')] = 2\gamma(\tau - \tau'), \quad (47)$$

which allows for the computation of ζ once $\gamma(\tau)$ is known. In the presence of dipolar interactions, Eq. (43) depends on $\gamma(\tau)$ and one has to resort to a numerical approach if one wishes to solve it self-consistently. However, if one simply takes the zeroth-order result $f(\omega) \approx -(e^2/2)|\omega|$, coming from the expansion of $f(\omega)$ in powers of $\gamma(\tau)$, one sees that, apart from the contribution coming from the line tension σ , Eqs. (44)–(46) are identical to the ones obtained by Hazareesing and Mézard [9] in their study of the roughening properties of a liquid-gas interface in a disordered solid. Also, if one expands the last term of Eq. (23) to quadratic order in $x(\tau) - x(\tau')$, one can see more directly that the domain-wall energy has, in this order of approximation, the same analytical form as that obtained by Kolomeiskii *et al.* for a domain wall in a ferroelastic material and as that used by Hazareesing and Mézard in the study of the problem referred above (provided one neglects the contribution of the

line tension in our case). Therefore, if one takes this approximation, one can, following Robbins and Joanny and also Kolomeiskii *et al.* [32,12], use an Imry-Ma type of argument [3] to show that the expected value of ζ is 1/3, this behavior of $\gamma(\tau)$ being valid for lengths L much larger than the Larkin length [33] ξ , which is given in our problem by $\xi \sim e^4 \mathcal{B}^3 / \tilde{\mu}^2 \Delta$. When $L \gg \xi$, we can disregard the line tension contribution to the action, since this energy term scales with L^{-1} . However, at short length scales compared to ξ , this term cannot be disregarded, since it gives an ω^2 contribution to the denominator of the Green's function appearing in Eq. (44). This contribution gives rise to a behavior $\gamma(\tau) \sim \tau$, at length scales $\leq \mathcal{B}$, as can be seen from the Mézard-Parisi solution [7,34]. Having in mind such behavior of $\gamma(\tau)$, we can write Eq. (43) as

$$f(\omega) = -e^2 \int_0^\xi d\tau [1 - \cos(\omega\tau)] \left[\frac{1}{2\pi\gamma(\tau)} - \frac{\tau}{4\sqrt{\pi}\gamma^{3/2}(\tau)} e^{\tau^{2/4}\gamma(\tau)} \text{erfc}\left(\frac{\tau}{2\gamma^{1/2}(\tau)}\right) \right] - e^2 \int_\xi^\infty d\tau [1 - \cos(\omega\tau)] \left[\frac{1}{2\pi\gamma(\tau)} - \frac{\tau}{4\sqrt{\pi}\gamma^{3/2}(\tau)} e^{\tau^{2/4}\gamma(\tau)} \text{erfc}\left(\frac{\tau}{2\gamma^{1/2}(\tau)}\right) \right]. \quad (48)$$

The integrand in the first term is regular at small τ [due to the behavior of $\gamma(\tau)$ at small τ] and its contribution is unimportant if $\omega\xi \ll 1$, which is the set of wave vectors that determines the roughening exponent ζ . In the second term, we can use, if $\mathcal{B} \ll \xi$, the asymptotic expansion for $\text{erfc}(x)$. We obtain, omitting the regular contribution of the first term,

$$f(\omega) \approx -\frac{e^2}{\pi} \int_\xi^\infty d\tau [1 - \cos(\omega\tau)] \left[\frac{1}{\tau^2} - \frac{6\gamma(\tau)}{\tau^4} + \dots \right], \quad (49)$$

where the dots stand for terms of order $\gamma^n(\tau)/\tau^{2n+2}$, with $n \geq 2$. Now, since $\gamma(\tau) \sim \tau^{2\zeta}$ with $\zeta < 1$, then the second term in Eq. (49) is much smaller than the first one and the omitted terms are of higher order. We can, therefore, keep only the first term and we obtain $f(\omega) \approx -(e^2/2)|\omega|$, i.e., the zeroth-order approximation for $f(\omega)$ is a good one. The validity of this approximation of course relies on the fact that $\zeta < 1$, but, as we shall show below, the result obtained indeed coincides with the Imry-Ma argument, i.e., $\zeta = 1/3$. If we now substitute this result for $f(\omega)$ in Eq. (46) we can perform the integral (with $\Omega_0 = 0$) and obtain a closed equation for $q(u)$. This equation reads

$$q(u) = \left(\frac{\tilde{\mu}^4 \Delta^2 \sigma^{1/2}}{(k_B T)^3} \right)^{1/3} r^{1/2}(u) \left[\frac{e^2}{4\sqrt{\sigma}} \frac{r^{1/2}(u)}{[q](u)} - \text{arctanh}\left(\frac{4\sqrt{r(u)\sigma}}{e^2}\right) \right]^{-1/3}, \quad (50)$$

where $r(u) = (e^4/16\sigma) - [q](u)$. Notice that $r(u)$ is positive at small u since $[q](u) \rightarrow 0$ when $u \rightarrow 0$. If $r(u)$ becomes negative, this equation is still valid, provided that we write $r^{1/2}(u) = i|r(u)|^{1/2}$ and analytically continue Eq. (50) to imaginary values. This equation relates $[q](u)$ to $q(u)$, with $[q]'(u) = uq(u)$ by definition, through a transcendental function and cannot be easily solved numerically. It reduces to the equation discussed by Mezárd and Parisi [7] if one takes the value of the dipolar interaction to zero (i.e., $e^2 = 0$) and to the equation discussed by Hazareesing and Mezárd [9] if one takes the line tension to zero (i.e., $\sigma = 0$). In these two cases, this equation reduces to trivial algebraic equations, which can be easily solved. In the first case, one obtains $\zeta = 1$, which coincides with the exact result of Zhang [6] and in the second case one obtains $\zeta = 1/3$, as pointed out above. In our case, we can nevertheless obtain information regarding the behavior of $q(u)$ at small u directly from Eq. (50). Performing some algebraic manipulations with Eq. (50) and using the identity $\operatorname{arctanh}(x) = \ln[(1-x)/(1+x)]$, one can show that

$$\lim_{u \rightarrow 0} [q](u)q^{-3}(u) = \frac{4(k_B T)^3}{\tilde{\mu}^4 \Delta^2 e^2}, \quad (51)$$

which shows that, since $[q](u) \rightarrow 0$ when $u \rightarrow 0$, then one must have $q(u) = Au^{1/2} + O(u^{1/2})$ for small u where $O(u^{1/2})$ indicates terms of order higher than $1/2$ and $A = (\tilde{\mu}^4 \Delta^2 e^2 / 12(k_B T)^3)^{1/2}$. Likewise, one must have $[q](u) = (1/3)Au^{3/2} + O(u^{3/2})$, where $O(u^{3/2})$ indicates terms of order higher than $3/2$. This coincides with the result obtained by Hazareesing and Mezárd [9], although their solution is valid in a finite interval around 0. In order to obtain a differential equation valid $\forall u \in [0, 1]$, we differentiate Eq. (50) with respect to u . We get

$$q'(u) = 0 \quad (52)$$

or

$$r(u) = -\frac{1}{2}uq(u) + \frac{e^2 u r^2(u)}{12\sqrt{\sigma}[q]^2(u)} \left(\frac{\tilde{\mu}^4 \Delta^2 \sigma^{1/2}}{(k_B T)^3} \right)^{1/3} \\ \times \left[\frac{e^2}{4\sqrt{\sigma}} \frac{r^{1/2}(u)}{[q](u)} - \operatorname{arctanh} \left(\frac{4\sqrt{r(u)\sigma}}{e^2} \right) \right]^{-4/3}.$$

Now, the second equation can, with the aid of Eq. (50), be written in the following form:

$$[q]^3(u) = \nu[q]^2(u) + \frac{1}{2}uq(u)\{[q]^2(u) - \eta q^3(u)\}, \quad (53)$$

with $\nu \equiv e^4/16\sigma$ and $\eta \equiv e^2(k_B T)^3/6\tilde{\mu}^4 \Delta^2 \sigma$. This equation is also valid when $r(u) < 0$ without the need to perform an analytic continuation, which is an advantage with regard to Eq. (50). Now, the physical dimensions of ν and η are, according to these definitions, given by $[\nu] = \text{j m}^{-3}$ and $[\eta] = \text{j}^{-1} \text{m}^3$ (in mks units). Furthermore, since S_0 has the di-

mensions of an energy and both $x(\tau)$ and τ have the dimensions of a length, it is also easy to see that the dimensions of $q(u)$ are the same as those of ν . If we define $z(u) = \int_0^u dv q(v)$, one has $q(u) = dz/du$ and $[q](u) = u(dz/du) - z(u)$. Substituting these equations in Eq. (53), one obtains a differential algebraic equation for $z(u)$. This equation reads

$$\eta u \left(\frac{dz}{du} \right)^4 + \left(u \frac{dz}{du} - z \right)^2 \left(u \frac{dz}{du} - 2(z + \nu) \right) = 0. \quad (54)$$

Since $z(u)$ and ν have the same dimensions, it is useful to write $z(u)$ in terms of a dimensionless function $z(u) = \nu x^2 l(x)$, where $x = 4u^{3/4}/3\alpha^{1/4}$ is a scaling variable with $\alpha = \eta\nu = e^6(k_B T)^3/96\tilde{\mu}^4 \Delta^2 \sigma^2$ being a dimensionless constant [35]. In terms of $l(x)$, the condition (51) becomes $l(0) = 1/2^{5/2}$, since the x^2 power corresponds to a $u^{3/2}$ behavior at small u . Substituting $z(u)$ in terms of $l(x)$ in Eq. (54) we obtain the following differential algebraic equation for $l(x)$:

$$[xl'(x) + 2l(x)]^4 + \left(\frac{3}{4}xl'(x) + \frac{1}{2}l(x) \right)^2 \left(\frac{3}{4}x^3l'(x) - \frac{1}{2}x^2l(x) - 2 \right) = 0, \quad (55)$$

which does not contain any dimensional parameters. The advantage of this equation with regard to Eq. (53) or Eq. (54) is its scaling form, which means that once we have obtained the scaling function $l(x)$, one simply has to substitute x by its expression in terms of u to obtain $z(u)$ for all ν and α . Furthermore, since we have to solve Eq. (54) for $u \in [0, 1]$, this means that we need to solve Eq. (55) for $x \in [0, 4/3\alpha^{1/4}]$. When α is large [36], this interval becomes a small interval around 0 and we can linearize Eq. (55), by writing $l(x) = 1/2^{5/2} + m(x)$. Substituting this equation above and keeping only terms linear in $m(x)$ and $m'(x)$, we obtain the differential equation

$$-\left(\frac{x}{8\sqrt{2}} + \frac{3x^3}{512} \right) m'(x) + \left(\frac{1}{4\sqrt{2}} - \frac{3x^2}{256} \right) m(x) = \frac{x^2}{1024\sqrt{2}}, \quad (56)$$

with solution

$$m(x) = \frac{8Bx^2 - 128x^2 \ln x - 3\sqrt{2}x^4}{8(32\sqrt{2} + 3x^2)^2}, \quad (57)$$

where B is an arbitrary constant. Since the solution $q'(u) = 0$ can be valid in a subinterval of $[0, 1]$, the general solution of Eq. (42) is

$$q(u) = \begin{cases} z'(u) & \text{if } u < u_c, \\ z'(u_c) & \text{if } u \geq u_c, \end{cases} \quad (58)$$

where $z(u) = \nu x^2 l(x)$ with $l(x)$ being given by the full solution of Eq. (55) in the general case or simply by the linear approximation

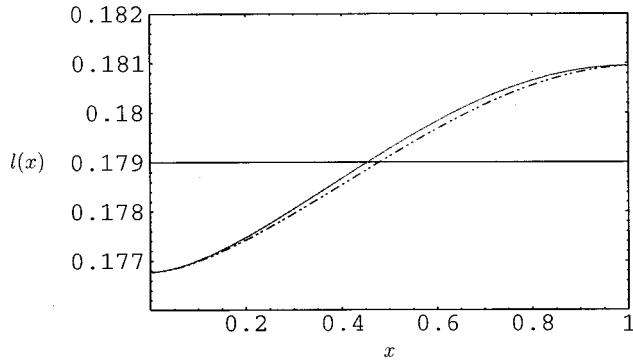


FIG. 1. Linearized (continuous line) and numerical (dashed line) solutions of Eq. (55) for the same initial (and arbitrary) condition $l(1)=0.18095$. In the case of the numerical solution, the MANPAK algorithm was iterated down to $x=0$. It is seen that the two solutions converge to one another close to $x=0$.

$$l(x) = \frac{1}{2^{5/2}} + \frac{8Bx^2 - 128x^2 \ln x - 3\sqrt{2}x^4}{8(32\sqrt{2} + 3x^2)^2}$$

in the case of large α . The values of the arbitrary integration constant B appearing in $l(x)$ and of u_c can be determined by substituting the expression for $q(u)$ obtained from $z(u)$ in Eqs. (42) and (50). We have solved Eq. (55) numerically using the MANPAK algorithm [37], available at netlib.org. A comparison between a polynomial fit of the numerical solution and the linearized solution (57) is shown in Figs. 1 and 2, which shows the correctness of this solution at small x . However, even in the case of large α , where the linear approximation can be used, we still need to solve the system composed by the transcendental Eqs. (42) and (50), and one would have to resort to a numerical approach. Nevertheless, since the behavior of $\gamma(\tau)$ at large length scales is solely determined by the behavior of $[q](u)$ at small u , it is still possible to show, following Hazareesing and Mézard [9], that the roughening exponent $\zeta = 1/3$. In order to do that, one has to notice that $[q](u) = 1/2 \nu x^2 l(x) + 3/4 \nu x^3 l'(x)$ for $u \leq u_c$, $[q](u) = [q](u_c)$ for $u > u_c$. Substituting this result for

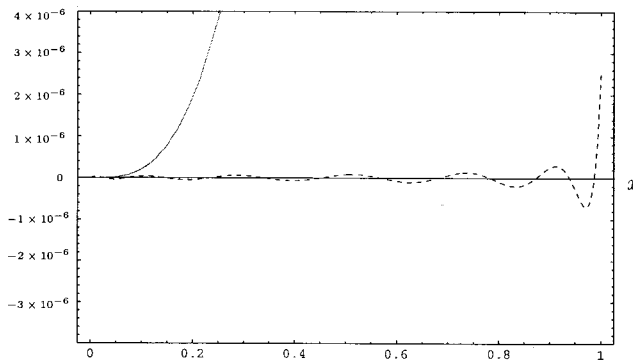


FIG. 2. Left-hand side of Eq. (55) evaluated for the linearized (continuous line) and numerical (dashed line) solutions. It is seen that the linearized solution gives better results close to $x=0$ but that it deviates significantly for larger values. The numerical solution is seen to oscillate around zero. Note that the scale of the plot is 10^{-6} .

$[q](u)$ in Eq. (44), we obtain for small ω , using $l(0) = 1/2^{5/2}$ and $l'(0) = 0$, which follow from the linear approximation, the result

$$\tilde{g}(\omega) = C[\sigma\omega^2 + (e^2/2)|\omega|]^{-5/3} + \dots, \quad (59)$$

where the dots indicate terms that diverge less strongly at small ω and where C is a numerical constant. From this result and from the definition of $\gamma(\tau)$, one immediately concludes that $\zeta = 1/3$, in agreement with the Imry-Ma argument of Robbins and Joanny. The determination of the crossover behavior of $\gamma(\tau)$, at length scales comparable with ξ as well as the determination of ξ itself (the Imry-Ma argument just gives the order of magnitude) requires, as stated above, the solution of the system of transcendental equations (42) and (50).

VI. CONCLUSIONS

We have obtained an expression for the elastic energy of a Néel domain wall in a thin ferromagnetic film in the presence of dipolar interactions and a quenched random field, beyond the quadratic approximation for the dipolar energy. Using the replica trick and a variational ansatz, we have obtained a set of self-consistent equations for the Green's functions of the displacement field of the domain wall. These equations were solved analytically in the case of two-dimensional dipolar interactions by making a quadratic approximation for the dipolar energy, which was justified on the basis of the different behavior of the domain wall at different length scales. The problem is then analogous to the one of a domain wall in a ferroelastic as studied by Kolomeiskii *et al.* and to the one of a liquid-gas contact line in a disordered solid, as studied by Hazareesing and Mézard. We, therefore, obtain a value $\zeta = 1/3$ for the roughening exponent of the domain wall.

From these calculations, we have obtained some important results. First, we were able to represent the dipolar interaction in the domain-wall Hamiltonian as the interaction of the quantum particle with an annealed gauge field. Also, in $1+1$ dimensions, one can integrate out such a field, leaving us with a polaronlike quantum Hamiltonian. Second, we have shown that a generalized replica symmetry breaking ansatz allows for the treatment of this problem. Note that this generalization leads to relatively simple equations in our case, since the nondiagonal part of the self-energy in replica space is still wave-vector independent, which is due to the short-range correlated nature of the random field. The treatment of long-range correlated disorder is, on the other hand, a much more complicated problem [38]. Finally, we were able to justify that taking the quadratic approximation for two-dimensional dipolar interactions is sufficient to obtain the correct value of the roughening exponent within this approximation.

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APPENDIX: COUPLING CONSTANTS AND THEIR PHYSICAL UNITS

Here we collect the definitions of the different coupling constants used in this paper, together with their dimensions in the mks system, denoted by $[\]$. One uses the units joule ($j = \text{kg m}^2 \text{ s}^{-2}$), ampere (A), and meter (m). The “fundamental” physical constants are the magnetic permittivity of the vacuum μ_0 , the Bohr magneton μ_B , the Landé factor for the system g_L , the numerical value of the spin S , the strength of the exchange interaction between nearest neighbors J (given in j), the amplitude of the random field correlations Δ (given in $\text{A}^2 \text{ m}^{-2}$) and the lattice constant a . Furthermore, in the case of a SC/FM/SC heterostructure, one also needs the London penetration depth of the superconductor λ_L and its relative permittivity μ_r . From the units of these quantities in the mks [39], one then determines the dimension of the coupling constants appearing below.

$$\sigma = \frac{J \pi^2 S^2}{a} \quad [\sigma] = j \text{ m}^{-1}, \quad (\text{A1})$$

$$\tilde{\mu} = \frac{\mu_0 \mu_B g_L \pi S}{a} \quad [\tilde{\mu}] = j \text{ A}^{-1}, \quad (\text{A2})$$

$$e_{2d}^2 = \frac{\mu_0 \mu_B^2 g_L^2 \pi^2 S^2}{2 \mu_r \lambda_L a^4} \quad [e_{2d}^2] = j \text{ m}^{-2}, \quad (\text{A3})$$

$$e_{3d}^2 = \frac{\mu_0 \mu_B^2 g_L^2 \pi^2 S^2}{2 a^4} \quad [e_{3d}^2] = j \text{ m}^{-1}, \quad (\text{A4})$$

$$\nu = \frac{e_{2d}^4}{16 \sigma} \quad [\nu] = j \text{ m}^{-3}, \quad (\text{A5})$$

$$\eta = \frac{e_{2d}^2 (k_B T)^3}{6 \tilde{\mu}^4 \Delta^2 \sigma} \quad [\eta] = j^{-1} \text{ m}^3, \quad (\text{A6})$$

where e_{2d}^2 is the value of the coupling constant for two-dimensional dipolar interactions (the case treated above) and e_{3d}^2 is the value for three-dimensional dipolar interactions.

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- [25] This mapping appears when considering the energy of linelike defects in the presence of long-range forces. See H. Kleinert, *Gauge Fields in Condensed Matter* (World Scientific, Singapore, 1989), Vol. I, and references therein.
- [26] This gauge transformation corresponds to adding the total derivative term $\int_{-L_y/2}^{L_y/2} d\tau (d/d\tau) \int_0^{x(\tau)} dx' A(x', \tau)$ to the action. Due to the periodic boundary conditions along the τ (y) direction the overall contribution of this term is zero. Also, notice that this gauge transformation can only be performed in one space dimension.
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- [29] This form of the correlator allows for a smooth introduction of a cutoff in the momentum integrals over q_x and leads naturally to the existence of a Larkin length. Note that a cutoff is always provided by the existence of a finite lattice spacing a , but its introduction via the noise correlations makes the equations analytically more tractable. In the limit in which one wants to treat isotropic correlations, one should take $\mathcal{B} = a$.
- [30] Note that in our case, we have chosen the disorder correlator $\Delta(q, \omega) = \Delta(q)$, which implies that the disorder is locally correlated along the $\tau(y)$ axis. Goldschmidt has also considered the case $\Delta(q, \omega) = \Delta(q) \delta_{\omega,0}$, which implies that the disorder correlator is independent of $\tau(y)$. For the corresponding problem of a quantum particle in a disordered potential, this just means that such potential is independent of time.
- [31] In the case of the problem of a quantum particle in a static random potential, Goldschmidt has considered an ansatz in which the only component of $Q_{ab}(\tau - \tau')$ that has a nondiagonal part is the zero-frequency one, i.e., Q_{ab}^0 .
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- [34] Here, we are assuming that the line tension plays a role at length scales smaller than \mathcal{B} whereas the dipolar interaction is unimportant, and that at length scales $\gg \xi$, the dipolar interaction is relevant whereas the line tension is not. One should first notice that there is also a competition between the dipolar term and the line tension term and that the dipolar term becomes dominant over the line tension term at length scales of the order $\sim \sigma/e^2$ (see Appendix B for units). Therefore, one must have that $\mathcal{B} \leq \sigma/e^2 \leq \xi = e^4 \mathcal{B}^3 / \tilde{\mu}^2 \Delta$, in order for the above assumption to be true, which of course depends on the values of \mathcal{B} and Δ we choose. If such condition holds, then one expects three distinct regimes. One regime for which $|\tau - \tau'| \leq \mathcal{B}$, in which thermal fluctuations are dominant since the random field does not play a role at these length scales. Here the roughening exponent is simply the thermal one, i.e., $\zeta = 1/2$, due to the line tension. A second regime $\mathcal{B} \leq |\tau - \tau'| \leq \xi$, in which one can use a linear approximation for the random-field and in which the replica symmetric solution holds. The value of ζ is also equal to $1/2$ in this regime, which follows from the analysis of Hazareesing and Mézard. Finally, if $\tau - \tau' \geq \xi$, $\zeta = 1/3$, since at these length scales the line tension does not play a role and the replica symmetry breaking solution of Hazareesing and Mézard holds.
- [35] Obviously, these results are valid provided that ν and η are not zero (i.e., $e \neq 0$). If $e = 0$ then one obtains from Eq. (53) the solution of Mézard and Parisi [7].
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- [38] As stated above, one very important example of long-range correlated disorder is the one considered by Goldschmidt that has a direct application to the problem of a quantum particle in a static random potential and to the related problem of a polymer in a medium with long-range correlated disorder. Since Goldschmidt has considered a rather general variational ansatz where the diagonal part of the self-energy is dependent on ω , as we have done above, such variational ansatz could in principle be used to treat the problem of a quantum polaron in a random static potential. We thank M. Mézard for calling our attention to this point.
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